FUNDAMENTAL RELATIONS ACROSS A STRONG STEADY SHOCK WAVE GIVING RISE TO A DISCONTINUITY IN CONDUCTIVITY

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1. The phenomena accompanying the motion of shock waves in media with infinity conductivity [1,2], are sufficiently well known. These phenomena have an important significance in astrophysical problems, in which the magnetic Reynolds' numbers in front of the shock waves attain very large values because of the extremely large characteristic linear dimensions. In papers [1,2] it is assumed that in front of and behind the shock waves the electric field achieves equilibrium with the induction field:

$$\mathbf{E} = -\frac{\mathbf{v} \times \mathbf{H}}{c_0} \tag{1.1}$$

Here $\mathbf{E}$ and $\mathbf{H}$ are the intensities of the electric and magnetic fields; $\mathbf{v}$ and c_0 are the material velocity and the velocity of light. Accordingly there are no currents outside the wave.

In aerodynamics, high speeds lend interest to the consideration of strong shock waves in media in which the conductivity in front of the shock wave is zero, whilst behind the wave it achieves high values as a result of the ionization caused by the wave. These phenomena occur, for example, in the motion of a body at high speed in the earth's atmosphere or in the motion of strong shock waves in pipes in the presence of an electromagnetic field. In these cases there are no currents in front of the shock wave even if condition (1.1) is violated, since the conductivity in front of the shock wave is equal to zero.

It is well known that ideal insulators do not exist in nature, and any gas is always to some extent ionized and consequently possesses a certain conductivity. The phenomena considered below occur in those cases when the magnetic Reynolds' numbers in front of and behind the shock wave satisfy the conditions

$$R_{m1} = \frac{4\pi\sigma_1 U_1 L_1}{c_0^2} \ll 1, \qquad R_{m2} = \frac{4\pi\sigma_2 U_2 L_2}{c_0^2} \gg 1$$

where U_1 and U_2, L_1 and L_2, σ_1 and σ_2 are respectively the characteristic velocities, linear dimensions and conductivities before and after the shock wave. We notice that here too an important assumption is made concerning the smallness of the inertial forces in front of the shock wave, i.e.

$$\frac{H_1^2}{4\pi\rho_1 U_1^2} R_{m1} \ll 1$$

With these assumptions we consider the motion of a strong stationary shock wave in a gas with a given electromagnetic field in front of the wave.

We shall assume that there are no currents in front of the shock wave on account of the small conductivity, whilst the electric and magnetic fields may be arbitrary. In the wave itself, due to the ionization, the conductivity is considerably increased and gives rise to currents which, according to our assumptions, rapidly die out; far behind the shock wave the electric and magnetic fields are connected by condition (1.1).

In order to obtain the connection between the parameters of the gas and the field at the passage of such a wave, let us consider the equations of motion of a gas in the presence of an electromagnetic field. The gas will be assumed inviscid and non-heat conducting, and radiation will not be taken into account. The coordinate system moves with the wave itself, and the x-axis is perpendicular to the wave.

Under these assumptions the equations of conservation of mass, momentum and energy can be written down respectively in the form [1]

$$\operatorname{div} \rho \mathbf{v} = 0, \qquad \frac{\partial \pi_{ik}}{\partial x_k} = 0, \qquad \operatorname{div} \mathbf{g} = 0 \tag{1.2}$$

Here ρ and p are the density and pressure of the gas, $\mathbf{v}$ is the velocity of the gas, π_{ik} is the tensor of the density of the stream momentum, $\mathbf{g}$ is the vector of the density of the stream energy:

$$\pi_{ik} = \rho v_i v_k + p\delta_{ik} - \frac{1}{4\pi}\left[H_i H_k - \frac{1}{2} H^2 \delta_{ik}\right] \tag{1.3}$$

$$\mathbf{g} = \rho \mathbf{v}\left(\frac{V^2}{2} + h\right) + \frac{c_0}{4\pi}\,\mathbf{E} \times \mathbf{H} \tag{1.4}$$

where h is the specific enthalpy of the gas. To the equations of motion of the gas (1.2) it is necessary to add also Maxwell's equations for the electromagnetic field

$$\operatorname{div}\mathbf{H}=0, \qquad \operatorname{rot}\mathbf{E}=0 \tag{1.5}$$

$$\operatorname{rot}\mathbf{H}=\frac{4\pi}{c_0}\mathbf{j}, \qquad \operatorname{div}\mathbf{E}=4\pi\rho_e \tag{1.6}$$

Here $\mathbf{j}$ is the current density, ρ_e is the charge density $|\epsilon = \mu = 1|$.

The Equations (1.2), (1.5) and (1.6) must be satisfied whatever the value of the conductivity and therefore are applicable both on the left and on the right of the shock wave. Applying Equations (1.2)-(1.5) to the shock wave, we obtain

$$[\rho v_x]=0, \qquad \left[p+\rho v_x^2+\frac{H^2}{8\pi}\right]=0, \qquad [H_x]=0 \tag{1.7}$$

$$\left[\rho v_x v_y-\frac{H_x H_y}{4\pi}\right]=0, \qquad \left[\rho v_x v_z-\frac{H_x H_z}{4\pi}\right]=0 \tag{1.8}$$

$$\left[\rho v_x\left(\frac{V^2}{2}+h\right)+\frac{c_0}{4\pi}(E_y H_z-E_z H_y)\right]=0 \ [E_y]=0, \tag{1.9}$$

$$[E_z]=0 \tag{1.10}$$

Here and in what follows the square brackets carry the significance that $[A] = A_1 - A_2$, where A_1 is the value of A in front of the wave, whilst A_2 is that behind the wave. Equations (1.6) determine the density of the surface current and charge at the shock wave and the change in the components of the electric field normal to the shock wave. These quantities do not enter the other equations and will not be considered further in this paper. Making use of condition (1.1) behind the shock wave, the relation (1.10) can be written in the form

$$c_0E_{1y}=v_{2x}H_{2z}-v_{2z}H_{2x}, \qquad -c_0E_{1z}=v_{2x}H_{2y}-v_{2y}H_{2x} \tag{1.11}$$

For the shock waves under consideration the dissociation and ionization of the gas are considerable, so for determination of the specific enthalpy behind the shock wave it is necessary to take account of these processes. In the general case it can be assumed that

$$h=h(p,\rho) \tag{1.12}$$

where h is a known function of pressure and density. Equations (1.7)-(1.9) and (1.11) form a closed system for determining the parameters of the gas and the field behind the wave. This system of equations differs from those considered in the papers [1,2] relating to a discontinuity in a medium with infinity conductivity, in that the electric field in front of the wave is assumed to be independent of the magnetic field.

2. In certain cases the System (1.7)-(1.9), (1.11), (1.12) admits of simplification.

1. When $v_{1x} = v_{2x} = 0$ (tangential discontinuity) we have the motion of a nonconducting medium relative to one with infinite conductivity. In this case the following conditions are fulfilled

$$\left[p + \frac{H^2}{8\pi}\right] = 0, \qquad H_x [H_y] = 0, \qquad H_x [H_z] = 0$$
$$E_y [H_z] = E_z [H_y], \qquad c_0 E_y = - v_{2z} H_x, \qquad c_0 E_z = v_{2y} H_x \tag{2.1}$$

If $H_x = 0$, then also $E_y = E_z = 0$. Moreover the magnetic field and the velocity permit an arbitrary jump on passing through the discontinuity. If $H_x \neq 0$, then $[H_y] = 0$, $[H_z] = 0$, $[p] = 0$. Choosing the coordinate system so that there is no electric field tangential to the discontinuity, we find that $v_{2z} = v_{2y} = 0$, whilst the velocity in the nonconducting medium can be arbitrary. The conditions (2.1) must be fulfilled in flow of a medium possessing infinite conductivity past a nonconducting body.

2. When $H_x \neq 0$ we can always choose a coordinate system such that the electric field tangential to the wave vanishes. In this coordinate system:

$$[\rho v_x] = 0, \qquad \left[p + \rho v_x^2 + \frac{H^2}{8\pi}\right] = 0, \qquad [H_x] = 0$$
$$\left[\rho v_x v_y - \frac{H_x H_y}{4\pi}\right] = 0, \qquad \left[\rho v_x v_z - \frac{H_x H_z}{4\pi}\right] = 0, \qquad \left[\frac{V^2}{2} + h\right] = 0 \tag{2.2}$$
$$v_{2x} H_{2z} = v_{2z} H_x, \qquad v_{2x} H_{2y} = v_{2y} H_x$$

From the two last equations it follows that behind the shock wave the velocity vector is collinear with the vector of the magnetic field intensity.

The coordinate system can always be oriented in such a way that the following condition is fulfilled:

$$(\rho v_x) v_{1z} - \frac{H_x}{4\pi} H_{1z} = 0 \tag{2.3}$$

In fact, let the angle formed by the vector $\mathbf{v}_{1\tau}$ with $\mathbf{H}_{1\tau}$ be equal to θ, whilst the angle formed by $\mathbf{H}_{1\tau}$ with the axis of z is equal to ϕ ($\mathbf{v}_{1\tau}$ and $\mathbf{H}_{1\tau}$ are the projections of $\mathbf{v}_1$ and $\mathbf{H}_1$ on the plane $x = 0$). Then the condition (2.3) can be expanded in the form

$$(\rho v_x) v_{1\tau} \cos(\varphi + \theta) - \frac{H_x}{4\pi} H_{1\tau} \cos\varphi = 0$$

Hence

$$\cos^2\varphi = \frac{(\rho v_x)^2 v_{1\tau}^2 \sin^2\theta}{(\rho v_x)^2 v_{1\tau}^2 \sin^2\theta + [(\rho v_x) v_{1\tau} - H_x H_{1\tau} / 4\pi]^2}$$

In the new coordinate system behind the wave the following relations apply:

$$(\rho v_x)\, v_{2z} - \frac{H_x}{4\pi} H_{2z} = 0, \qquad H_x v_{2z} - v_{2x} H_{2z} = 0$$

From this system it follows that either

$$(\rho v_x)\, v_{2x} = H_x^2/4\pi$$

and v_{2z} and H_{2z} may be different from zero, or else

$$(\rho v_x)\, v_{2x} \neq H_x^2/4\pi$$

and $v_{2z} = H_{2z} = 0$.

In the first case the velocity normal to the wave behind the shock wave will be equal to the Alfven velocity. It is not difficult to show that in this case

$$\frac{v_{1y}}{H_{1y}} = \frac{v_{1z}}{H_{1z}}$$

i.e. in front of the shock wave the velocity vector and the vector of magnetic field intensity lie in a plane, normal to the wave, whilst behind the wave the velocity vector may depart from this plane. This case is analogous to Alfven's wave.

In this case the vector of velocity behind the shock wave lies in the plane $z = 0$.

3. With $H_x = 0$ it follows from Equation (1.9) that the velocity tangential to the wave is continuous. The coordinate system can therefore be chosen so that it is, in general, zero; the axis of y is directed along the electric field tangential to the wave. In this coordinate system the relations at the shock wave can be written down in the form*

$$[\rho v_x] = 0, \qquad \left[p + \rho v_x^2 + \frac{H^2}{8\pi}\right] = 0, \left[\rho v_x \left(\frac{V^2}{2} + h\right) + \frac{c_0}{4\pi}\,(E_y H_z - E_z H_y)\right] = 0 \qquad (2.4)$$

$$c_0 E_y = v_{2x} H_{2z}, \qquad H_{2y} = 0, \qquad h = h\,(p, \rho)$$

This case is analogous to the perpendicular wave considered in [1].

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* An analogous system of equations is considered in the paper [3], published after the submission of the present paper to the press. The paper [3] also contains the Hugoniot adiabatic analysis for the case in which, as proved in Section 3 of the present paper, a stationary wave does not exist.

3. In the foregoing paragraphs it was assumed that the interaction of a strong shock wave in a gas with an electromagnetic field could be reduced to a stationary wave. However, for an arbitrarily specified electromagnetic field in front of the shock wave, in spite of the existence of a solution of the System (1.7)-(1.9), (1.11), (1.12), satisfying the condition of increase of entropy, a stationary wave is not always realized. In order to demonstrate this, let us consider the following example. Let

$$v_{1x} \neq 0, \qquad H_{1y} \neq 0, \qquad E_{1z} = H_x = 0$$

As conductivity only makes its appearance in the shock wave, then according to Ohm's law the current arising is

$$\mathbf{j} = \sigma \left[\mathbf{E} + \frac{\mathbf{V} \times \mathbf{H}}{c_0}\right]$$

Making use of Maxwell's equations and assuming that inside the wave all quantities depend only on x, we have

$$\nu_m \frac{dH_y}{dx} = v_x H_y \qquad \left(\nu_m = \frac{c_0^2}{4\pi\sigma}\right), \qquad \text{or} \qquad H_y(x) = H_{1y} \exp \int_{-\infty}^{x} \frac{v_x\, dx}{\nu_m}$$

Since $v_x/\nu_m > 0$ when $x \to \infty$, then it follows that

$$\frac{H_y(x)}{H_{1y}} \to \infty \qquad \text{as } x \to \infty$$

i.e. in this case there does not exist a steady solution of the equations of the structure of the shock wave with $H_{1y} \neq 0$, in spite of the fact that there exists a solution of the System (2.4).

A similar state of affairs occurs in the case when

$$E_y \neq 0, \qquad v_{1x} \neq 0, \qquad H_x = H_{1z} = 0$$

In this case the inductive field always increases the original electric field when $v_x > 0$, and therefore the electric currents do not die out in the shock wave as $x \to \infty$.

The examples given prove that the electromagnetic field in front of the shock wave must satisfy certain special conditions. These conditions can be obtained only from consideration of the equations of the structure of the shock wave.

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